AN INVARIANCE PRINCIPLE FOR THE LAW OF THE ITERATED LOGARITHM FOR ADDITIVE FUNCTIONALS OF MARKOV CHAINS

GUANGYU YANG AND YU MIAO

Abstract. We prove the Strassen's strong invariance principle for vector-valued additive functionals of a Markov chain via the martingale argument and the theory of fractional coboundaries. The hypothesis is a moment bound on the resolvent.

1. Introduction

Let $(X_n)_{n\geq 0}$ denote a stationary ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a measurable space $(\mathcal{X}, \mathcal{B})$. Let $Q(x, dy)$ be its transition kernel and π the stationary initial distribution. Furthermore, for $p \geq 1$, let $L^p(\pi)$ denote the space of (equivalence classes of) $\mathcal{B}$-measurable functions $g : \mathcal{X} \to \mathbb{R}^d$ for some $d \geq 1$ and such that $\|g\|_p^p := \int_{\mathcal{X}} |g(x)|^p \pi(dx) < \infty$, and let $L_0^p(\pi)$ denote the set of $g \in L^p(\pi)$ for which $\int_{\mathcal{X}} g d\pi = 0$. Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Now, fix d and an $\mathbb{R}^d$-valued function $g \in L_0^2(\pi)$. For $n \geq 0$, define

$$S_{n+1} = S_{n+1}(g) := \sum_{i=0}^{n} g(X_i) \text{ and } S_0 = 0. \tag{1.1}$$

For the question of central limit type results for S_n, there have been numerous studies from many angles and under different assumptions; see Maxwell and Woodroofe [10], Derriennic and Lin [5] [6] and references therein. In this note, we mainly consider the iterated logarithm type results for S_n. Since the appearance of Strassen's paper [18], almost sure invariance principles for the law of iterated logarithm have been obtained for a large class of independent and dependent sequence $(Y_n)_{n\geq 1}$; see Strassen [19], Hall and Heyde [8], and Philipp and Stout [16]. Here, the Skorokhod representation plays an important role.

It is well known that, the law of the iterated logarithm (in short LIL) is closely related to the central limit theorem (in short CLT) in some sense. There are several approaches to these problems. If the chain is Harris recurrent, then the problems may be reduced to the independent case in a certain sense, see Meyn and Tweedie [11] and Chen [3]. If there exists a solution to Poisson's equation, $h = g + Qh$, then the LIL and CLT problems may be reduced to the martingale case, see also Gordin and Lifsic [7] and Meyn and Tweedie [11]. Bhattacharya [2] obtained the functional

Date: September 20, 2006; revised November 13, 2006.

2000 *Mathematics Subject Classification.* Primary 60F17, secondary 60J10, 60J55.

Key words and phrases. Invariance principle; the law of the iterated logarithm; additive functionals of Markov chains; Dunford-Schwartz operator; fractional coboundaries; vector-valued martingales.

CLT and LIL for ergodic stationary Markov processes by discussing the infinitesimal generator. Wu [20] extended the forward-backward martingale decomposition of Meyer-Zheng-Lyons's type from the symmetric case to the general stationary situation and gave the Strassen's strong invariance principle.

Our goal, in this paper, is to consider the problem that S_n satisfies the LIL under some proper conditions. Note that, Rassoul-Agha and Seppäläinen [17] mainly relied on the invariance principle for vector-valued martingales, so it is likely to obtain the invariance principle for LIL for the vector-valued additive functionals of a Markov chain, only if we can develop the corresponding theory for vector-valued martingales. However, we encounter the essential difficulties, when considering the vector-valued martingale, since Monrad and Philipp [13] proved that it is impossible to embed a general $\mathbb{R}^d$-valued martingale in an $\mathbb{R}^d$-valued Gassian process. For the strong approximation of random sequence taking values in general Banach space, please refer to Philipp [15] and the references given there.

In the present paper, we will take along the lines of Kipnis and Varadhan [9] and Maxwell and Woodroofe [10], to the case where a solution is not required. Moreover, we identify the lim sup in LIL just the square root of the trace of the diffusion matrix corresponding to the functional CLT.

Let us explain the organization of this paper. In Section 2, we state our main results. Section 3 gives the proof of our main results mentioned in Section 2, which mainly depends on the strong approximation of vector-valued martingales (see Berger [1]), and the theory of fractional coboundaries developed by Derriennic and Lin [4].

2. **Main results**

For a function $h \in L^1(\pi)$, and π-a.e. $x \in \mathcal{X}$ define an operator

$$Qh(x) = \int h(y)Q(x, dy). \tag{2.1}$$

Obviously, Q is a contraction on $L^p(\pi)$ for $p \geq 1$. For $\varepsilon > 0$, let h_ε be the solution of the equation

$$(1 + \varepsilon)h = Qh + g.$$

In fact,

$$h_\varepsilon = \sum_{n=1}^{\infty} (1 + \varepsilon)^{-n} Q^{n-1} g. \tag{2.2}$$

Note that $h_\varepsilon \in L^p(\pi)$, if $g \in L^p(\pi)$. Let π_1 be the joint distribution of X_0 and X_1, so that $\pi_1(dx_0, dx_1) = Q(x_0, dx_1)\pi(dx_0)$; denote the L^2-norm on $L^2(\pi_1)$ by $\| \cdot \|_1$; and let

$$H_\varepsilon(x_0, x_1) = h_\varepsilon(x_1) - Qh_\varepsilon(x_0)$$

for x_0, $x_1 \in \mathcal{X}$. For any $\varepsilon > 0$, let

$$M_n(\varepsilon) = \sum_{i=0}^{n-1} H_\varepsilon(X_i, X_{i+1}) \text{ and } R_n(\varepsilon) = Qh_\varepsilon(X_0) - Qh_\varepsilon(X_n),$$

hence, by the simple computation,

$$S_n(g) = M_n(\varepsilon) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon). \tag{2.3}$$

For convenience, we summarize the results of Maxwell and Woodroofe [10] as the following theorem.

Theorem MW *Assume that $g \in L_0^2(\pi)$ and that there exists an $\alpha \in (0, 1/2)$ such that*

$$\|\sum_{i=0}^{n-1} Q^i g\|_2 = O(n^\alpha). \tag{2.4}$$

Then we have

(1) *The limit $H = \lim_{\varepsilon\to 0^+} H_\varepsilon$ exists in $L^2(\pi_1)$. Moreover, if one defines*

$$M_n = \sum_{i=0}^{n-1} m_i,$$

where $m_i = H(X_i, X_{i+1})$, then $(m_n)_{n\geq 0}$ is a stationary and ergodic $\mathbb{P}$-square integrable martingale difference sequence, with respect to the filtration $\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n)\}_{n\geq 0}$;

(2) *$\|h_\varepsilon\|_2 = O(\varepsilon^{-\alpha})$, and if $R_n = S_n - M_n = M_n(\varepsilon) - M_n + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon)$, then*

$$\mathbb{E}(|R_n|^2) = O(n^{2\alpha}).$$

Remarks 2.1. *Furthermore, if there exists a $p > 2$ such that $g \in L^p(\pi)$, then there exists a $q \in (2, p)$ such that $H \in L^q(\pi)$ and $(M_n)_{n\geq 1}$ is an L^q-martingale; see the Theorem 1 of Rassoul-Agha and Seppäläinen [17].*

For introducing our main results, we need give a few more notations. Let $C([0, 1], \mathbb{R}^d)$ be the Banach space of continuous maps from $[0, 1]$ to $\mathbb{R}^d$, endowed with the supremum norm $|||\cdot|||$, using the Euclidean norm in $\mathbb{R}^d$. Denote K the set of absolutely continuous maps $f \in C([0, 1], \mathbb{R}^d)$, such that

$$f(0) = 0, \qquad \int_0^1 |\dot{f}(t)|^2 dt \leq 1,$$

where, $\dot{f}$ denotes the derivative of f determined almost everywhere with respect to Lebesgue measure. Obviously, K is relatively compact and closed.

Define

$$\xi_n(t) = (2n \log\log n)^{-1/2}[S_k + (nt - k)g(X_k)]$$

for $t \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 0, 1, 2, \cdots, n-1$. In order to avoid difficulties in specification, we adopt the convention that $\log\log x = 1$, if $0 < x \leq e^e$. Then, ξ_n is a random element with values in $C([0, 1], \mathbb{R}^d)$.

After these preparations, we are now in a position to state our main results.

Theorem 2.2. *Let $g \in L_0^p(\pi)$ $(p > 2)$ and assume that there exists an $\alpha \in (0, 1/2)$ for which (2.4) is satisfied. Then, the sequence of functions $(\xi_n(\cdot), n \geq 1)$ is relatively compact in the space $C([0, 1], \mathbb{R}^d)$, and the set of its limit points as $n \to \infty$, coincides with $\sqrt{\mathrm{tr}(\mathfrak{D})}K$, where $\mathrm{tr}(\cdot)$ denotes the trace operator of a matrix and*

$\mathfrak{D} = \mathbb{E}(M_1 M_1^t) = \int HH^t d\pi_1$ is the diffusion matrix corresponding to the functional central limit theorem; see Rassoul-Agha and Seppäläinen [17].

Theorem 2.3. *Let $g \in L_0^p(\pi)\,(p > 2)$ and assume that there exists an $\alpha \in (0, 1/2)$ for which (2.4) is satisfied. Then*

$$\limsup |S_n| / \sqrt{2n \log\log n} = \sqrt{\operatorname{tr}(\mathfrak{D})} \quad \mathbb{P} - a.s. \tag{2.5}$$

Remarks 2.4. *In fact, $\operatorname{tr}(\mathfrak{D}) = \|H\|_1^2$. And particularly, if putting $d = 1$, we can obtain the main results of Miao and Yang [12].*

Remarks 2.5. *For $n \geq 0$, define $S_n^* = S_n - \mathbb{E}_{X_0} S_n$, since the Theorem 3 of Rassoul-Agha and Seppäläinen [17], the above Theorem 2.1 and Theorem 2.2 also hold for S_n^*.*

3. Proof of main results

3.1. Proof of Theorem 2.2.

Proof. For $0 \leq t \leq 1$, define

$$\zeta_n(t) = (2n \log\log n)^{-1/2} M_{[nt]},$$
$$\eta_n(t) = (2n \log\log n)^{-1/2} B(nt),$$

where, M_n is as defined in Section 2 and $B(\cdot)$ is an $\mathbb{R}^d$-valued Brownian motion with mean 0 and diffusion matrix $\mathfrak{D}$. Theorem 1 of Strassen [18] shows that $(\eta_n(\cdot))_{n\geq 1}$ is relatively compact and the set of its limit points coincides with $\sqrt{\operatorname{tr}(\mathfrak{D})}K$.

Notice that by the part (1) of Theorem MW, $(M_n)_{n\geq 1}$ is a square integrable martingale with strictly stationary increments. Moreover,

$$\mathbb{E}(< u, m_0 >^2) < \infty \;\text{ and }\; \mathbb{E}(< u, m_0 >) = 0, \;\text{ for all } u \in \mathbb{R}^d, \tag{3.1}$$

where, $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^d$. Therefore, Corollary 4.1 of Berger [1] implies that,

Without changing its distribution, one can redefined the sequence $(M_n)_{n\geq 1}$ on a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ on which there exists an $\mathbb{R}^d$-valued Brownian motion $(B(t))_{t\geq 0}$ with mean 0 and diffusion matrix $\mathfrak{D}$ such that

$$|M_{[t]} - B(t)| = o((t \log\log t)^{-1/2}), \; \hat{\mathbb{P}} - a.s. \; (as\ t \to \infty). \tag{3.2}$$

where, $\mathfrak{D} = \lim_{n\to\infty} n^{-1} Cov(M_n)$.

Remarks 3.1. *Birkhoff-Khinchin's ergodic theory and together with the simple calculation shows that, $\mathfrak{D} = \mathbb{E}(M_1 M_1^t) = \int HH^t d\pi_1$, is the diffusion matrix corresponding to the functional central limit theorem; see also Rassoul-Agha and Seppäläinen [17].*

That is to say,

$$\sup_{0\leq t\leq 1} |M_{[nt]} - B(nt)| = o((2n \log\log n)^{-1/2}), \quad \mathbb{P} - a.s.$$

Hence,

$$\begin{aligned}|||\zeta_n - \eta_n||| =&(2n\log\log n)^{-1/2}\sup_{0\le t\le 1}|M_{[nt]} - B(nt)|\\ =&o(1), \qquad\qquad \mathbb{P}-a.s.\end{aligned}$$

Define

$$\tilde{\zeta}_n(t) = (2n\log\log n)^{-1/2}[M_k + (nt-k)m_k]$$

for $t \in [\frac{k}{n}, \frac{k+1}{n}),\ k = 0, 1, 2, \cdots, n-1$. Then $\tilde{\zeta}_n \in C([0,1], \mathbb{R}^d)$ and

$$\sup_{t\in[0,1)}|\zeta_n(t) - \tilde{\zeta}_n(t)| = (2n\log\log n)^{-1/2}\max_{0\le k\le n-1}|m_k|.$$

Next, we give the order estimation of $\max_{0\le k\le n-1}|m_k|$.

Lemma M *(See Móricz [14]) Let $p > 0$ and $\beta > 1$ be two positive real numbers and Z_i be a sequence of random variables. Assume that there are nonnegative constants a_j satisfying*

$$\mathbb{E}|\sum_{j=1}^{i} Z_j|^p \le (\sum_{j=1}^{i} a_j)^\beta, \tag{3.3}$$

for $1 \le i \le n$. Then

$$\mathbb{E}(\max_{1\le i\le n}|\sum_{j=1}^{i} Z_j|^p) \le C_{p,q}(\sum_{i=1}^{n} a_i)^\beta, \tag{3.4}$$

for some positive constant $C_{p,\beta}$ depending only on p and β.

Lemma 3.2. *For any enough large n, there exists a positive constant C such that*

$$\mathbb{E}(\max_{1\le i\le n}|m_i|^q) \le C\mathbb{E}|m_1|^q. \tag{3.5}$$

Proof. Since the part (1) of Theorem MW and Remarks 2.1, let $\mathbb{E}|m_1|^q = a^2(q)$ and for any $k \ge 1$, we have the following relations,

$$\mathbb{E}|m_k|^q \le (\sum_{i=1}^{k} a_i)^2, \tag{3.6}$$

where $a_1 = a(q)$ and $a_i = 0$ for $2 \le i \le k$. Hence, by Lemma M, there exists a constant $C > 0$, such that

$$\mathbb{E}(\max_{1\le i\le n}|m_i|^q) \le C(\sum_{i=1}^{n} a_i)^2 = C\mathbb{E}(|m_1|^q). \tag{3.7}$$

This completes the proof of the lemma. □

For any $\epsilon > 0$, Lemma 3.2 immediately yields,

$$\mathbb{P}(\max_{0\le k\le n-1}|m_k| \ge \epsilon(2n\log\log n)^{1/2}) = O((n\log\log n)^{-q/2}).$$

By Borel-Cantelli's lemma, we have

$$(2n\log\log n)^{-1/2}\max_{0\le k\le n-1}|m_k| = o(1), \qquad \mathbb{P}-a.s. \tag{3.8}$$

Hence,

$$|||\zeta_n - \tilde{\zeta}_n||| = \sup_{t\in[0,1)} |\zeta_n(t) - \tilde{\zeta}_n(t)| = o(1), \qquad \mathbb{P} - a.s.$$

The above discussions immediately yield the following claim:

$(\tilde{\zeta}_n(\cdot), n \geq 1)$ *is relatively compact and the set of its limit points coincides with* $\sqrt{\operatorname{tr}(\mathfrak{D})}K$.

Now, we turn to deal with the neglectable term R_n in the sense of functional LIL. Firstly, let us recall the concept of Dunford-Schwartz (DS) operator; see Derriennic and Lin [4]. We call T a DS operator on L^1 of a probability space, if T is a contraction of L^1 such that $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L^\infty$. If θ is a measure preserving transformation in a probability space (Ω, Σ, μ), then the operator $Tf = f \circ \theta$ is a DS operator on $L^1(\mu)$. More generally, any Markov transition operator P with an invariant probability measure yields a positive DS operator.

Lemma DL *(See Derriennic and Lin [4])*

(1) *Let T be a contraction in a Banach space X, and let $0 < \beta < 1$. If* $\sup_n \|\frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k y\| < \infty$, *then $y \in (I-T)^\alpha X$ for every $0 < \alpha < \beta$.*

(2) *Let T is a DS operator in $L^1(\mu)$ of a probability space, and fix $1 < p < \infty$, with dual $q = p/(p-1)$. Let $0 < \alpha < 1$, and $f \in (I-T)^\alpha L^p$. If $\alpha > 1 - \frac{1}{p} = \frac{1}{q}$, then* $\frac{1}{n^{1/p}} \sum_{k=0}^{n-1} T^k f \to 0$ *a.e.*

To apply the above lemma, i.e., the theory of fractional coboundaries named by Derriennic and Lin [4], we need construct a DS operator. On $\mathcal{X} \times \mathcal{X}$, define

$$f(x_0, x_1) = g(x_0) - H(x_0, x_1),$$

then

$$\begin{aligned} R_n =& S_n - M_n \\ =& \sum_{i=0}^{n-1} [g(X_i) - H(X_i, X_{i+1})] \\ =& \sum_{i=0}^{n-1} f(X_i, X_{i+1}). \end{aligned} \tag{3.9}$$

Let θ be the shift map on the path space $\mathcal{X}^{\mathbb{N}}$ for the Markov chain which is a contraction on $L^2(\mathbb{P})$. Hence, θ is a DS operator. For a sequence $x = (x_i)_{i\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$, define $F(x) = f(x_0, x_1)$, then we have

$$F \in L^2(\mathbb{P}) \quad \text{and} \quad R_n = \sum_{k=0}^{n-1} F \circ \theta^k.$$

From the part (2) of Theorem MW, there exists a constant $1/2 < \beta < 1-\alpha$, such that

$$\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=0}^{n-1} F \circ \theta^k \right\| < \infty, \tag{3.10}$$

Since the part (1) of Lemma DL and $0 < \alpha < 1/2$, we have $F \in (I-\theta)^\eta L^2(\mathbb{P})$, for some $\eta \in (1/2, 1-\alpha)$. By the part (2) of Lemma DL, we have

$$\frac{1}{n^{1/2}} R_n \to 0, \qquad \mathbb{P}-a.s.$$

Furthermore, applying an elementary property of real convergent sequences, we immediately get

$$\max_{0\le k\le n} |R_k| = o((2n\log\log n)^{1/2}), \quad \mathbb{P}-a.s.$$

Consequently,

$$(2n\log\log n)^{-1/2} \sup_{0\le t\le 1} |R_{[nt]}| \to 0, \quad \mathbb{P}-a.s. \tag{3.11}$$

From above discussions, we complete the proof of Theorem 2.2. □

3.2. **Proof of Theorem 2.3.**

Proof. Here, we take along the lines of the proof of Theorem 4.8 in Hall and Heyde [8]. Let $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. For any $\mathbb{R}^d$-valued function f, denote $f = (f_1, f_2, \cdots, f_d)^t$. By the definition of K, we have, for any $f \in \sqrt{tr(\mathfrak{D})}K$,

$$\begin{aligned} |f(t)|^2 &= \sum_{i=1}^d (\int_0^t \dot{f}_i(s)ds)^2 \\ &\le \sum_{i=1}^d (\int_0^t \dot{f}_i(s)^2 ds) \int_0^t 1ds \le tr(\mathfrak{D})t \end{aligned} \tag{3.12}$$

where, the first inequality by the Cauchy-Schwartz's inequality. So, $|f(t)| \le \sqrt{tr(\mathfrak{D})t}$. It follows that $\sup_{t\in[0,1]} |f(t)| \le \sqrt{tr(\mathfrak{D})}$. Hence, by Theorem 2.1,

$$\limsup \sup_{t\in[0,1]} |\xi_n(t)| \le \sqrt{tr(\mathfrak{D})}, \quad \mathbb{P}-a.s. \tag{3.13}$$

and setting $t = 1$,

$$\limsup |S_n|/\sqrt{2n\log\log n} \le \sqrt{\mathrm{tr}(\mathfrak{D})}, \quad \mathbb{P}-a.s. \tag{3.14}$$

On the other hand, we put $f(t) = t\sqrt{\frac{tr(\mathfrak{D})}{d}} \sum_{i=1}^d e_i$, $t \in [0,1]$. Then, $f \in \sqrt{tr(\mathfrak{D})}K$ and so for $\mathbb{P}-a.s.\,\omega$, there exists a sequence $n_k = n_k(\omega)$, such that

$$\xi_{n_k}(\cdot)(\omega) \xrightarrow{|||\cdot|||} f(\cdot). \tag{3.15}$$

Particularly, $f(1) = \sqrt{\frac{tr(\mathfrak{D})}{d}} \sum_{i=1}^d e_i$, $|\xi_{n_k}(1)(\omega)| \longrightarrow |f(1)|$. That is to say,

$$|S_{n_k}|/\sqrt{2n_k\log\log n_k} = \sqrt{\mathrm{tr}(\mathfrak{D})}, \quad \mathbb{P}-a.s. \tag{3.16}$$

This completes the proof of Theorem 2.3. □

Acknowledgements

The authors wish to thank Prof. Y. Derriennic and Prof. M. Lin for sending the key reference paper [4].

References

[1] Berger, E. (1990). An almost sure invariance principle for stationary ergodic sequences of Banach space valued random variables. *Probab. Th. Rel. Fields* **84**, 161-201.

[2] Bhattacharya, R. N. (1982). On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrs. verw. Geb.* **60**, 185-201.

[3] Chen, X. (1999). *Limit theorems for functionals of ergodic Markov chains with general state space.* Memor. Amer. Math. Soci. No. 664.

[4] Derriennic, Y. and Lin, M. (2001). Fractional Poisson equations and ergodic theorems for fractional coboundaries. *Israel J. Math.* **123**, 93-130.

[5] Derriennic, Y. and Lin, M. (2001). The central limit theorem for Markov chains with normal transition operators, started at a point. *Probab. Th. Rel. Fields* **119**, 508-528.

[6] Derriennic, Y. and Lin, M. (2003). The central limit theorem for Markov chains started at a point. *Probab. Th. Rel. Fields* **125**, 73-76.

[7] Gordin, M. I. and Lifsic, B. A. (1978). The central limit theorem for stationary Markov processes. *Dokl. Akad. Nauk SSSR* **19**, 392-394.

[8] Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its application.* Acad. Press, New York.

[9] Kipnis, C. and Varadhan, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104**, 1-19.

[10] Maxwell, M. and Woodroofe, M. (2000). Central limit theorems for additive functionals of Markov chains. *Ann. Probab.* **28** (2), 713-724.

[11] Meyn, S. P. and Tweedie, R. L. (1993). *Markov chains and stochastic stability.* Springer-Verlag, New York.

[12] Miao, Y. and Yang, G. (2006). The law of the iterated logarithm for additive functionals of Markov chains. *Submitted.*

[13] Monrad, D. and Philipp, W. (1990). The problem of embedding vector-valued martingales in Gaussian process. *Teor. Veroyatn. Primen.* **35**, 384-387.

[14] Móricz, F. (1976). Moment inequalities and the strong laws of large numbers. *Z. Wahrs. verw. Geb.* **35**, 299-314.

[15] Philipp, W. (1986). Invariance principles for independent and weakly dependent random variables. In: *Dependence in Probability and Statistics. Proc. Conf.* Oberwolfach 1985. Boston Basel Stuttgart: Birkhäuser.

[16] Philipp, W. and Stout, W. F. (1975). *Almost sure invariance principles for partial sums of weakly dependent random variables.* Memor. Amer. Math. Soci. No. 161.

[17] Rassoul-Agha, F. and Seppäläinen, T. (2004). An almost surely invariance principle for additive functionals of Markov chains. arXiv: math. PR/0411603.

[18] Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrs. verw. Geb.* **3**, 211-226.

[19] Strassen, V. (1965). Almost sure behavior of sums of independent random variables and martingales. *Pro. 5th Berkeley Symp. Math. Stat. and Probab.* **2**, 315-344.

[20] Wu, L. M. (1999). Forward-backward martingale decomposition and compactness results for additive functionais of stationary ergodic Markov processes. *Annals de l'i. H. P., Série Probab. and Stat.* **35** (2), 121-141.

School of Mathematics and Statistics, Wuhan University, 430072 Hubei, China
E-mail address: study_yang@yahoo.com.cn

School of Mathematics and Statistics, Wuhan University, 430072 Hubei, China and School of Mathematics and Information Science, Henan Normal University, 453007 Henan, China.
E-mail address: yumiao728@yahoo.com.cn